Knot diagram invariants and bounds for the number of Reidemeister moves needed for unknotting

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PRIMES Conference

May 21, 2016

Background

Knots are circles embedded in 3-dimensional space

Intuitively, a knot is a piece of string that has been tangled around itself with no self-intersection in 3D space with the ends connected to each other.

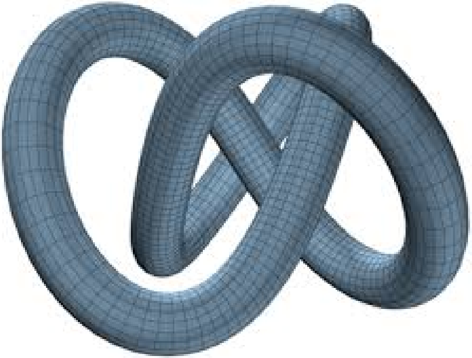


Figure 1: A knot.

Knots and links can be represented by diagrams

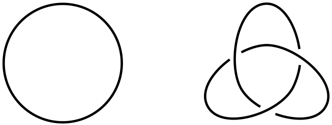


Figure 2: Diagrams of the trivial knot and trefoil knot.

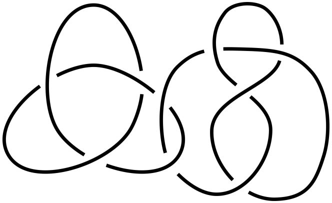


Figure 3: A diagram of a link.

Knots and links can be represented by diagrams

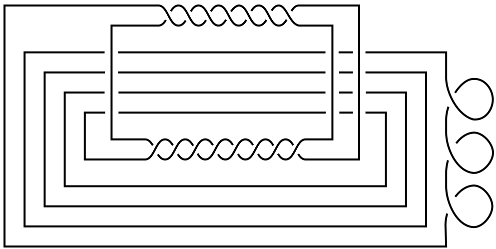


Figure 4: A much more complex diagram of the unknot

A knot or link can be assigned an orientation

Any component of a projection (i.e. any distinct curve) can be assigned a consistent direction moving along the curve, called an orientation. Any projection with orientations assigned is called **oriented**.

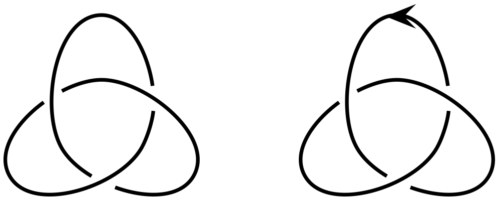


Figure 5: A non-oriented and oriented trefoil knot.

Reidemeister Moves can be performed on a projection to produce another projection of the same knot

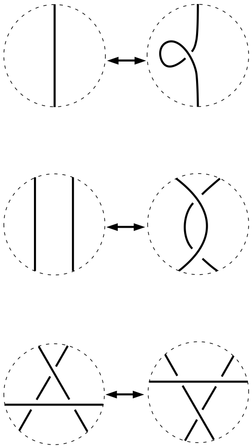


Figure 6: Reidemeister moves of Type I, II, and III

All projections of a knot are connected by a series of Reidemeister moves

Theorem (Reidemeister)

Diagrams D and E represent the same knot if and only if D and E are connected by a series of Reidemeister moves.

The Problem

Can we find good (or better) lower and/or upper bounds on the number of Reidemeister moves needed to transform one projection of a knot into a another. Particularly, can we use a knot diagram invariant to do so?

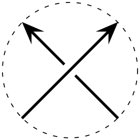
Motivation

- A solution to the problem has applications in finding an algorithm for determining a series of Reidemeister moves between diagrams;
- And more broadly, algorithmically distinguish knots

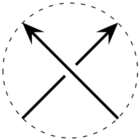
Diagram Invariants and Current Work

Writhe is the sum of signs of crossings

Assign each crossing of an oriented knot or link projection with a positive or negative 1, based on the picture below. For a crossing c, we call this number the sign of c, or $\mathrm{sgn}(c)$. The sum of signs among all crossings is called the **writhe**.



+1



-1

Figure 7: Positive and negative crossings of oriented knots

Writhe is the sum of signs of crossings

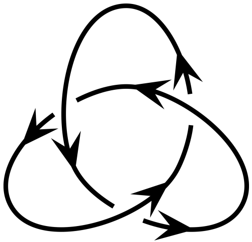


Figure 8: The writhe is $(-1) + (-1) + (-1) = -3$

Self-Crossing Index (SCI): Calculating Index

To calculate this diagram invariant, first, the index of a region must be defined. Set the exterior region to have index 0 and use the following rule:

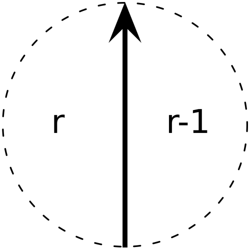
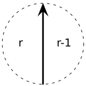


Figure 9: Rule for assigning index to regions of a diagram.

We can label all the regions of a knot with indices



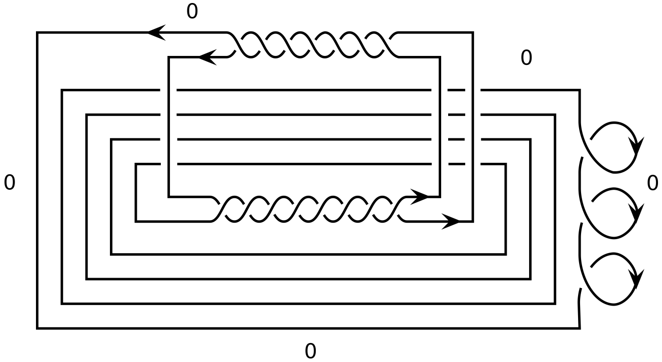
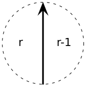


Figure 10: First assign an index of 0 to the exterior region.

We can label all the regions of a knot with indices



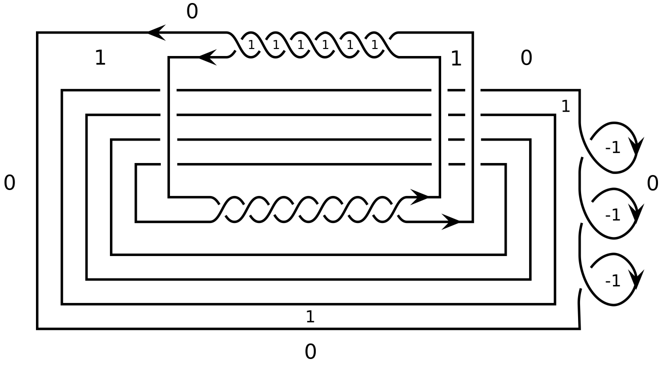
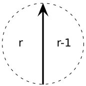


Figure 11: Then assign indices to the regions adjacent to the exterior region.

We can label all the regions of a knot with indices



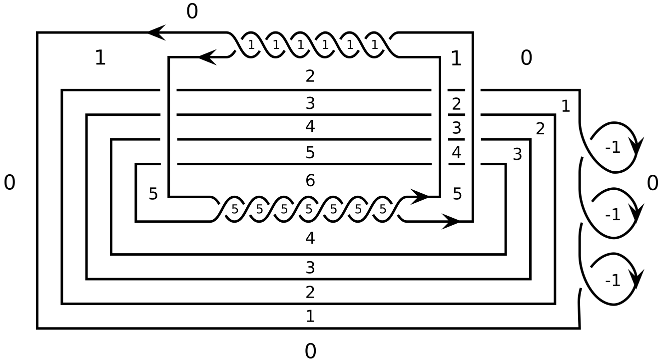
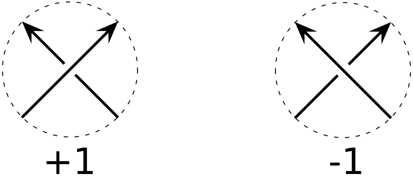
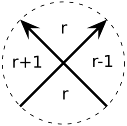


Figure 12: Diagram with all indices of regions assigned.

Self-Crossing Index (SCI) is a diagram invariant



$$\mathrm{sgn}(c) = \pm 1$$



$$\mathrm{qInd}(c) = \frac{(r+1) + r + r + (r-1)}{4} = r$$

$$\mathrm{SCI}(K) = \sum_{c \in C} \mathrm{sgn}(c)\,\mathrm{qInd}(c)$$

Self-Crossing Index (SCI)

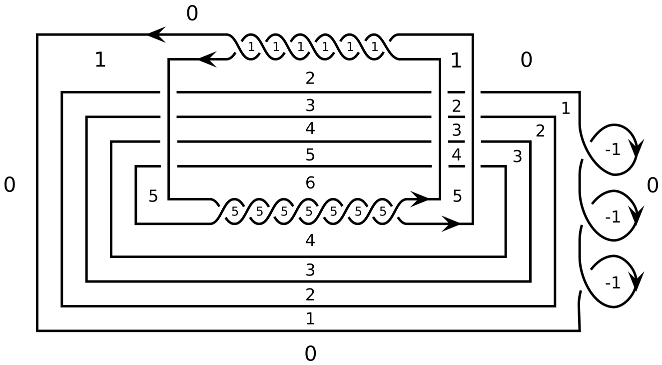


Figure 13: SCI is $(1)(7)(1) + (-1)(8)(5) + (-1)(2+3+4+5) + (1)(2+3+4+5) + (1)(1+2+3+4) = -23$

Changes to SCI under Reidemeister moves are well understood

Under Reidemeister Type 1 moves, SCI can change by any number.

Under Reidemeister Type 2 moves, SCI is unchanged.

Under Reidemeister Type 3 moves, SCI changes by ± 1.

For framed knot diagrams, SCI only changes under framed Reidemeister Type 3 moves

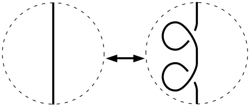


Figure 14: Under framed Reidemeister 1 moves, SCI remains unchanged.

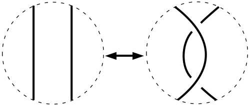


Figure 15: Under framed Reidemeister 2 moves, SCI remains unchanged.

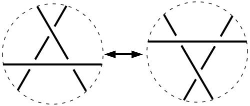


Figure 16: Under framed Reidemeister 3 moves, SCI changes by ± 1.

The difference in SCI is a lower bound to the number of framed Reidemeister moves between framed knot diagrams

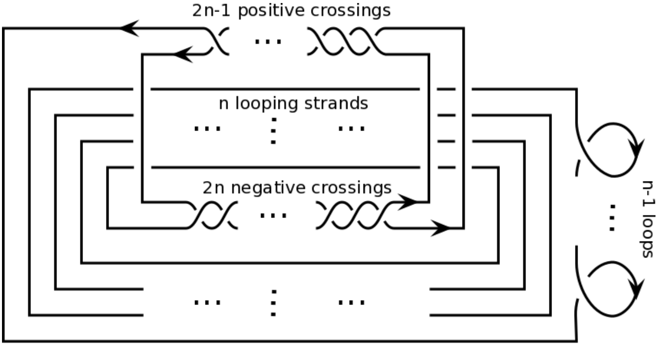


Figure 17: From Hass and Nowik, 2007

$$\mathrm{SCI}(D_n) = -\frac{1}{2}(3n^2 - n + 2)$$

Conclusion

Summary

Problem: Can we use knot diagram invariants to find good (or better) lower and/or upper bounds on the number of Reidemeister moves needed to transform one projection of a knot into a another?

- SCI is the sum of the product of the sign and index of each crossing of a diagram.
- SCI change predictably under framed Reidemeister moves.
- We have found a quadratic lower bound for a family of knots.

Future Work:

- Develop a diagram invariant or value of a knot that is useful for all knots.
- Is there a family of unknots that needs a polynomial of the number of crossings of at least degree 3 in terms of Reidemeister moves to get the trivial projection?
- Could bounds for framed unknots give bounds for unknots or for all knots in general?

Acknowledgments

I would like to thank:

- My mentor Piotr Suwara
- The MIT-PRIMES program
- Prof. Maciej Borodzik
- My parents